

Winding numbers and $SU(2)$ -representations of knot groups

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Abstract

Given an abelian group A and a Lie group G , we construct a bilinear pairing from $A \times \pi_1(\mathcal{R})$ to $\pi_1(G)$, where \mathcal{R} is a subvariety of the variety of representations $A \rightarrow G$.

In the case where A is the peripheral subgroup of a torus or two-bridge knot group, $G = S^1$ and \mathcal{R} is a certain variety of representations arising from suitable $SU(2)$ -representations of the knot group, we show that this pairing is not identically zero. We discuss the consequences of this result for the $SU(2)$ -representations of fundamental groups of manifolds obtained by Dehn surgery on such knots.

1 Introduction

The real algebraic variety of representations from a 3-manifold group $\pi_1(M)$ to $SU(2)$ or $SO(3)$ has long been a subject of interest, giving rise as it does to useful invariants such as the Casson invariant and the A -polynomial [3].

In the case where ∂M is a torus – in particular, where M is the exterior of a knot in S^3 – there is a particular interest in finding representations which vanish on a given slope $\alpha \in \mathbb{Q} \cup \{\infty\}$ on ∂M , and hence give rise to

a representation of $\pi_1(M(\alpha))$, where $M(\alpha)$ is the manifold obtained from M by Dehn filling along α .

A description of the character variety in the case of a 2-bridge knot is given by Burde in [1]. For twist knots, a more detailed description is given by Uygur and Azcan in [8].

Burde [1] used this description to show that nontrivial representations $\pi_1(M(+1)) \rightarrow SU(2)$ exist for any nontrivial 2-bridge knot exterior M , and deduced the Property P Conjecture for 2-bridge knots. More recently, Kronheimer and Mrowka [5] proved the Property P Conjecture in full by showing that nontrivial representations $\pi_1(M(+1)) \rightarrow SO(3)$ exist for an arbitrary nontrivial knot exterior M .

In another article [6], the same authors proved that there is an irreducible representation $\pi_1(M(r)) \rightarrow SU(2)$ (that is, a representation with nonabelian image), for any nontrivial knot exterior M and any slope $r \in \mathbb{Q}$ such that $|r| \leq 2$. One consequence of this (see [2, 4]) is that every nontrivial knot has a nontrivial A -polynomial.

In the present note, we construct a bilinear pairing $\pi_1(\mathcal{C}) \times \pi_1(\partial M) \rightarrow \mathbb{Z}$ for suitable subsets \mathcal{C} of the variety \mathcal{R} of representations $\pi_1 M \rightarrow SU(2)$, and apply it to Burde's description [1] of \mathcal{R} in the case of 2-bridge knots, to show that the restriction $|r| \leq 2$ in [6] can be weakened in this case:

Theorem 1.1 *Let M be the exterior of a nontrivial 2-bridge knot in S^3 which is not a torus knot, and let α be any non-meridian slope in ∂M . Then there exists an irreducible representation $\pi_1(M(\alpha)) \rightarrow SU(2)$.*

Since there are many examples of lens spaces obtainable by Dehn surgery on nontrivial knots, it is clear that the above theorem cannot possibly extend from 2-bridge knots to arbitrary knots. However, by varying the subset \mathcal{C} of the representation in our construction, we can adapt the technique to consider also reducible representations.

As an example, we prove the following result for torus knots.

Theorem 1.2 *Let X be the exterior of the (p, q) torus knot, where $1 < p < q$, and $X(\alpha)$ the manifold obtained from X by Dehn filling along a non-meridian slope $\alpha \in \mathbb{Q}$. Then*

- (1) *if $\alpha = pq$ and $p > 2$, then $\pi_1(X(\alpha))$ admits an irreducible representation to $SU(2)$;*

- (2) if $\alpha = pq$ and $p = 2$, then $\pi_1(X(\alpha))$ admits no irreducible representation to $SU(2)$, but admits a representation to $SO(3)$ with nonabelian image;
- (3) if $\alpha = pq \pm \frac{1}{n}$ for some positive integer n , then every representation from $\pi_1(X(\alpha))$ to $SO(3)$ has abelian image;
- (4) for any other value of α , $\pi_1(X(\alpha))$ admits an irreducible representation to $SU(2)$.

Results of [7] indicate that this result is in a sense best possible: for example, in Case (3) the Dehn surgery manifold $X(\alpha)$ is a lens space.

The paper is organised as follows. In Section 2 below we recall some basic properties of the $SU(2)$ representation and character varieties of a knot group. In Section 3 we describe our bilinear pairing, in a fairly general context. We then apply this in Sections 4 and 5 to prove Theorems 1.1 and 1.2 respectively.

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2 The $SU(2)$ representation and character varieties

If Γ is any finitely presented group, and G is a (real) algebraic matrix group, then the set of representations $\Gamma \rightarrow G$ forms a real affine algebraic variety \mathcal{R} on which G acts by conjugation, giving rise to a quotient *character variety* \mathcal{X} .

For the purposes of the present paper, Γ will always be a knot group, and $G = SU(2)$. In this case \mathcal{R} is naturally expressed as a union of two closed $SU(2)$ -invariant subvarieties $\mathcal{R}_{red} \cup \mathcal{R}_{irr}$, and hence also \mathcal{X} is a union of subvarieties $\mathcal{X}_{red} \cup \mathcal{X}_{irr}$. Here \mathcal{R}_{red} denotes the variety of *reducible* representations $\rho : \Gamma \rightarrow SU(2)$, in other words those for which the resulting Γ -module \mathbb{C}^2 splits as a direct sum of two 1-dimensional modules. This happens precisely when the image of ρ is abelian, in other words when ρ is induced from a representation of $\Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}$. Hence \mathcal{R}_{red} is canonically homeomorphic to $SU(2) \cong S^3$. The corresponding character subvariety \mathcal{X}_{red} is canonically homeomorphic to the closed interval $[-2, 2] \subset \mathbb{R}$, parametrised by the trace $Tr(\rho(\mu))$, where ρ is a representative of a conjugacy class of

reducible representations, and $\mu \in \Gamma$ is a fixed meridian element. The complement of \mathcal{R}_{red} in \mathcal{R} is not closed, but its closure is a subvariety \mathcal{R}_{irr} which is $SU(2)$ -invariant and hence gives rise to a closed subvariety \mathcal{X}_{irr} of \mathcal{X} .

Now fix once and for all a meridian $\mu \in \Gamma$, and consider the following subset \mathcal{C} of \mathcal{R} . A representation $\rho : \Gamma \rightarrow SU(2)$ belongs to \mathcal{C} if and only if

$$\rho(\mu) = \begin{pmatrix} x + iy & 0 \\ 0 & x - iy \end{pmatrix}$$

with $x, y \in \mathbb{R}$, $x^2 + y^2 = 1$ and $y \geq 0$. Note that every representation in \mathcal{R} is conjugate to one in \mathcal{C} , so the quotient map $\mathcal{R} \rightarrow \mathcal{X}$ restricts to a surjection on \mathcal{C} (and to a homeomorphism $\mathcal{C} \cap \mathcal{R}_{red} \rightarrow \mathcal{X}_{red}$).

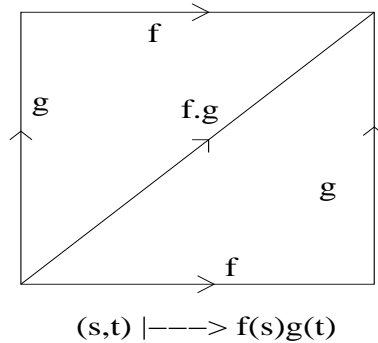
3 Winding numbers

Let A be an abelian group, G a (connected) Lie group, and \mathcal{D} a subset of the variety of representations $A \rightarrow G$. Given any path $P = \{\rho_t, 0 \leq t \leq 1\}$ in \mathcal{D} , and any $a \in A$, we obtain a path $P(a) = \{\rho_t(a), 0 \leq t \leq 1\}$ in G .

Clearly, if P' is homotopic (rel end points) to P , then $P'(a)$ is homotopic (rel end points) to $P(a)$, for any $a \in A$. Hence we obtain a pairing

$$\nu : A \times \pi_1(\mathcal{D}) \rightarrow \pi_1(G), \quad \nu(a, [P]) := [P(a)].$$

Remark Recall that, if $f, g : [0, 1] \rightarrow G$ are closed paths in the topological group G , based at the identity element $1 \in G$, then $[f][g] = [f \cdot g] = [g][f]$ in $\pi_1(G, 1)$, where $f \cdot g$ denotes the pointwise product $t \mapsto f(t)g(t) \in G$. This can easily be seen, for example, from the diagram below, representing the map $[0, 1]^2 \rightarrow G$, $(s, t) \mapsto f(s)g(t)$.



In particular, $\pi_1(G, 1)$ is abelian, so the above definition of ν is unaffected by base-point choices.

Proposition 3.1 *The pairing $\nu : (a, [P]) \mapsto [P(a)]$ defined above is bilinear.*

Proof. For a fixed element $a \in A$, if $P.Q$ is the concatenation of paths P, Q in \mathcal{D} , then $(P.Q)(a)$ is the concatenation of $P(a), Q(a)$ in G , so $[P] \mapsto [P(a)]$ is a homomorphism $\pi_1(\mathcal{D}) \rightarrow \pi_1(G)$.

For $a, b \in A$ and a fixed path P in \mathcal{D} , we have $P_t(ab) = P_t(a)P_t(b)$ for each $t \in [0, 1]$, since P_t is a representation $A \rightarrow G$. By the above remark, $[P(ab)] = [P(a)][P(b)]$ in $\pi_1(G)$. In other words, $a \mapsto [P(a)]$ is a homomorphism $A \rightarrow \pi_1(G)$. \square

We apply Proposition 3.1 in the following restricted context. Let X be the exterior of a nontrivial knot in S^3 , and let $A = \pi_1(\partial X) \cong \mathbb{Z}^2$. Let $\mu, \lambda \in A$ denote a fixed meridian and longitude respectively.

Let G be the Lie group $S^1 = \{z \in \mathbb{C}; |z| = 1\}$.

The subset \mathcal{D} of the variety of representations $A \rightarrow S^1$ arises as follows. We regard S^1 as the subgroup of $SU(2)$ consisting of diagonal matrices. Recall that \mathcal{R} is the variety of representations $\pi_1(X) \rightarrow SU(2)$, and that \mathcal{C} is the subvariety of \mathcal{R} consisting of representations $\rho : \pi_1(X) \rightarrow SU(2)$ such that $\rho(\mu)$ is diagonal, and the imaginary part of the $(1, 1)$ entry of $\rho(\mu)$ is non-negative. Since A is abelian and $\pi_1(X)$ is generated by conjugates of μ , it follows that $\rho(A)$ contains only diagonal matrices whenever $\rho \in \mathcal{C}$. We define \mathcal{D} to be the set of representations $A \rightarrow S^1$ that arise as restrictions of representations in \mathcal{C} .

Note that $\pi_1(S) \cong \mathbb{Z}$, so the bilinear pairing $\nu : A \times \pi_1(\mathcal{D}) \rightarrow \pi_1(S^1)$ is integer-valued.

Proposition 3.2 *For each $\gamma \in \pi_1(\mathcal{D})$ let $K_\gamma \subset A$ denote the kernel of the homomorphism $A \rightarrow \mathbb{Z}$, $a \mapsto \nu(a, \gamma)$. Then either $K_\gamma = A$ or $K_\gamma = \mathbb{Z}\mu$, the subgroup of A generated by μ .*

Proof. Certainly μ belongs to K_γ for all $\gamma \in \pi_1(\mathcal{D})$, since for $\rho \in \mathcal{C}$ we have $\rho(\mu)$ contained in an open interval in S^1 (so the winding number of $\rho(\mu)$ as ρ travels around C is zero).

On the other hand, let $c = \nu(\lambda, \gamma)$. Then by bilinearity, for any $m, n \in \mathbb{Z}$ we have $\nu(m\mu + n\lambda, \gamma) = cn$. If $cn = 0$ for some n then either $c = 0$ or $n = 0$. In the first case $\nu(m\mu + n\lambda, \gamma) = 0$ for all m, n . In the second case, $m\mu + n\lambda = m\mu \in \mathbb{Z}\mu$. \square

Corollary 3.3 *If the pairing $\nu : A \times \pi_1(\mathcal{D}) \rightarrow \mathbb{Z}$ is not uniformly vanishing, and α is any non-meridian slope on ∂X , then $\pi_1(X(\alpha))$ admits a nontrivial representation to $SU(2)$, where $X(\alpha)$ is the 3-manifold obtained from X by Dehn-filling along α .*

Proof. By hypothesis, $K_\gamma \neq A$ for some $\gamma \in \pi_1(\mathcal{D})$, so $K_\gamma = \mathbb{Z}\mu$ by the Proposition. Since $\alpha \notin \mathbb{Z}\mu$, it follows that $\nu(\alpha, \gamma) \neq 0$. Hence the map $S^1 \rightarrow S^1$ defined by $t \mapsto \gamma_t(\alpha)$, has nonzero winding number, and hence in particular is surjective. Thus we may choose $t \in S^1$ such that $\gamma_t(\alpha) = 1 \in S^1$. Now γ_t is the restriction of a nontrivial representation $\sigma : \pi_1(X) \rightarrow SU(2)$, so $\sigma(\alpha) = 1 \in SU(2)$ and hence σ induces a nontrivial representation

$$\tau : \pi_1(X(\alpha)) = \pi_1(X) / \langle\langle \alpha \rangle\rangle \rightarrow SU(2).$$

□

In practice, to find suitable closed paths in \mathcal{D} we may find a closed path in \mathcal{C} and project it to \mathcal{D} using the restriction map $\rho \mapsto \rho|_A$. The next result shows that it is equally valid to work in the character variety \mathcal{X} rather than \mathcal{C} .

Lemma 3.4 *The restriction map $\mathcal{C} \rightarrow \mathcal{D}$, $\rho \mapsto \rho|_A$, factors through \mathcal{X} .*

Proof. Given $\rho, \rho' \in \mathcal{C}$ with the same image in \mathcal{X} , we know that ρ, ρ' are conjugate by some matrix $M \in SU(2)$. If $\rho(\mu) \in Z(SU(2)) = \{\pm I\}$, then the image of ρ is central and so $\rho' = \rho$. Otherwise, $\rho(\mu) = \rho'(\mu)$ is a diagonal matrix with non-real diagonal entries, so the conjugating matrix M must also be diagonal. But in this case $\rho(A)$ consists only of diagonal matrices, which therefore commute with M , so the restrictions of ρ and ρ' to A coincide. □

An immediate consequence of Lemma 3.4 is that any path in \mathcal{R} between two conjugate representations gives rise to a closed path in \mathcal{D} by first projecting to \mathcal{X} and then applying the restriction map $\mathcal{X} \rightarrow \mathcal{D}$.

4 Two-bridge knots

In this section we prove the following result.

Theorem 4.1 *Let \mathcal{R}_{irr} be the variety of irreducible $SU(2)$ -representations of a two-bridge knot group G , and let A be a peripheral subgroup of G . Then there is a closed curve γ in \mathcal{R}_{irr} such that the pairing $\nu : \pi_1(\gamma) \times A \rightarrow \mathbb{Z}$ is not identically zero.*

Proof. A two-bridge knot group G has a presentation of the form

$$G = \langle x, y | Wx = yW \rangle,$$

where $W = W(x, y)$ is a word of the form $x^{\varepsilon(1)}y^{\varepsilon(2)} \dots y^{\varepsilon(2n)}$ with $\varepsilon(i) = \pm 1$ for each i . Here x and y are meridians. The symmetry of the presentation ensures that $xW^* = W^*y$ in G , where $W^*(x, y) := W(y, x)$. Hence $\beta = W^*W$ commutes with the meridian x , so is a peripheral element and represents a slope on the boundary torus of the knot exterior.

The exponents $\varepsilon(i)$ can be more explicitly described. There is an odd integer k coprime to $2n + 1$ such that

$$\varepsilon(i) = (-1)^{\lfloor \frac{ik}{2n+1} \rfloor}$$

for each i . In particular, since

$$\frac{ik}{2n+1} - 1 < \lfloor \frac{ik}{2n+1} \rfloor < \frac{ik}{2n+1}$$

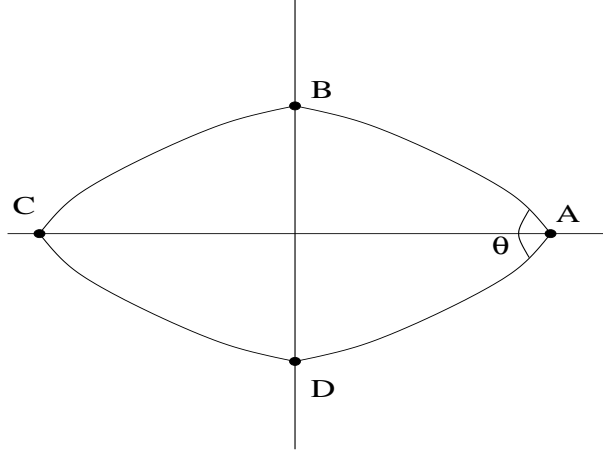
for each i , we have

$$\lfloor \frac{ik}{2n+1} \rfloor + \lfloor \frac{(2n+1-i)k}{2n+1} \rfloor = k - 1 \equiv 0 \pmod{2}$$

for each i , so that $\varepsilon(2n+1-i) = \varepsilon(i)$. From this, it follows that

$$W(x^{-1}, y^{-1}) = W(y, x)^{-1} = W^{*-1} \quad \text{and} \quad W^*(x^{-1}, y^{-1}) = W^{-1}.$$

The following construction is essentially due to Burde (see [1, p.116]). Under the action of $SU(2)$ by rotations on S^2 , we may choose fixed points of $\rho(x), \rho(W), \rho(y), \rho(W^*)$ as the vertices A, B, C, D respectively of a spherical rhombus, such that $\rho(W)(A) = C$ and $\rho(W^*)(C) = A$. (There are degenerate cases: possibly $A = C$ if $\rho(G)$ is abelian; possibly $B = D$ if $\rho(W) = \rho(W^*)$ with $\rho(W)^2 = -I$.) It follows that the angle of rotation of W^*W is 2θ modulo 2π , where θ is the angle \widehat{DAB} of the rhombus.



Conjugacy in $SU(2)$ allows us freedom to place this rhombus where we wish. Let us choose to place it with $A = (1, 0, 0)$, and $C = (\cos \psi, \sin \psi, 0)$ with $0 < \psi < \pi$.

If we have a path ρ_t ($0 \leq t \leq 1$) of representations, then this gives rise to a path $A_t B_t C_t D_t$ of rhombi, and a path $\theta_t \in \mathbb{R}/(2\pi\mathbb{Z})$ of corresponding angles. Parameters t with $\theta_t \in 2\pi\mathbb{Z}$ correspond to degenerate rhombi with $B_t = D_t$, and hence to representations ρ_t with $\rho_t(W) = \rho_t(W^*)$.

Among all $SU(2)$ representations of G , a special rôle is played by those whose image in $SO(3)$ is dihedral, in other words where $\rho(x)^2 = \rho(y)^2 = -I$. In this case, the points B, D of our rhombus coincide with the north and south poles $N, S = (0, 0, \pm 1)$. Burde [1, pp. 116-117] explains that, if Γ is the group of a two-bridge knot which is not a torus knot, then there is a path ρ_t of irreducible representations joining two dihedral representations ρ_0, ρ_1 , such that B, D switch poles on travelling from $t = 0$ to $t = 1$. In other words, the change of angle $\theta_1 - \theta_0$ on traversing this path is an odd multiple of 2π (in particular nonzero). Replacing the path ρ_t by a smooth approximation if necessary, we may assume that θ_t is differentiable as a function of t , and express this as

$$\int_0^1 \frac{\partial \theta_t}{\partial t} dt \neq 0.$$

Now consider another path of representations $\bar{\rho}_t$, defined by $\bar{\rho}_t(x) = -\rho_t(x^{-1})$, $\bar{\rho}_t(y) = -\rho_t(y^{-1})$. The equation $W(x^{-1}, y^{-1}) = W^{*-1}$ enables us to verify that $\bar{\rho}_t$ is indeed a representation for each t . Moreover, since

$\rho_t(x)^2 = \rho_t(y)^2 = -I$ for $t = 0, 1$, it follows that $\bar{\rho}_t = \rho_t$ for $t = 0, 1$. Finally, since $\bar{\rho}_t(W^*W) = \rho_t(W^*W)^{-1}$, the change in θ along the path $\bar{\rho}$ is the negative of the change along the path ρ_t :

$$\frac{\partial \bar{\theta}_t}{\partial t} = -\frac{\partial \theta_t}{\partial t}.$$

If γ is the closed curve formed by concatenating the paths ρ_t and $\bar{\rho}_{1-t}$, the change in θ around γ is precisely twice that along ρ_t , namely an odd multiple of 4π :

$$\int_{\gamma} \frac{\partial \theta_t}{\partial t} dt = \int_0^1 \frac{\partial \theta_t}{\partial t} dt + \int_1^0 \frac{\partial \bar{\theta}_t}{\partial t} dt = 2 \int_0^1 \frac{\partial \theta_t}{\partial t} dt \neq 0.$$

In particular $\nu([\gamma], W^*W) \neq 0$. \square

Corollary 4.2 *Let X be the exterior of a two-bridge knot in S^3 , and let $X(\alpha)$ be the manifold formed from X by Dehn filling along a non-meridian slope α in ∂X . Then $\pi_1(X(\alpha))$ admits an irreducible representation to $SU(2)$.*

Proof. By Theorem 4.1, there is a closed curve γ of irreducible representations $\pi_1(X) \rightarrow SU(2)$ such that the pairing ν on $\pi_1(\gamma) \times \pi_1(\partial X)$ is not identically zero.

Then $\nu([\gamma], -) : \pi_1(\partial X) \rightarrow \mathbb{Z}$ has kernel $\mu\mathbb{Z}$. Since $\alpha \notin \mu\mathbb{Z}$, $\nu([\gamma], \alpha) \neq 0$. In other words, the closed curve $t \mapsto \gamma_t(\alpha) \in S^1$ has non-zero winding number, and so is surjective. There exists a point $\rho \in \gamma$ such that $\rho(\alpha) = 1$ in $SU(2)$. Since $\pi_1(X(\alpha))$ is the quotient of $\pi_1(X)$ by the normal closure of α , ρ induces a representation $\pi_1(X(\alpha)) \rightarrow SU(2)$ with nonabelian image. \square

5 Torus knots

In this section we demonstrate that the pairing ν is not identically zero on suitable curves in the $SU(2)$ -representation variety of a torus knot. We then apply this to the fundamental group of any manifold obtained by nontrivial Dehn surgery on a torus knot, and study its representations to $SU(2)$.

The (p, q) -torus knot has fundamental group $\Gamma = \langle x, y | x^p = y^q \rangle$. In particular, it has nontrivial centre, generated by $\zeta = x^p = y^q$. If $\{\mu, \lambda\}$ is any meridian-longitude pair, then ζ belongs to the peripheral subgroup generated by $\{\mu, \lambda\}$, since it commutes with μ .

The character variety \mathcal{X} of $Hom(\Gamma, SU(2))$ splits into a number of arcs as follows. As for all knots, the subvariety \mathcal{X}_{red} corresponding to reducible representations is isomorphic to the closed interval $[-2, 2]$, parametrised by the trace of $\rho(\mu)$.

If $\rho : \Gamma \rightarrow SU(2)$ is an irreducible representation, then $\rho(x), \rho(y)$ are non-commuting matrices with $\rho(x)^p = \rho(y)^q$. This can arise only if $\rho(x)^p = \rho(y)^q = \pm I$, where I is the identity matrix. Hence $\rho(x)$ has trace $2 \cos(a\pi/p)$ and $\rho(y)$ has trace $2 \cos(b\pi/q)$ for some integers a, b of the same parity. There are $(p-1)(q-1)/2$ open arcs $A_{(a,b)}$ in the irreducible character variety, one corresponding to each pair (a, b) of integers with $1 \leq a \leq p-1$, $1 \leq b \leq q-1$, $a \equiv b$ modulo 2. Each open arc $A_{(a,b)}$ is the interior of a closed arc $\overline{A}_{(a,b)}$ in the whole character variety, whose endpoints are reducible characters.

Lemma 5.1 *The endpoints of $\overline{A}_{(a,b)}$ are the points*

$$2 \cos(c\pi/pq), 2 \cos(d\pi/pq) \in [-2, 2] \cong \mathcal{X}_{red},$$

where where $c, d \in \{1, \dots, pq-1\}$ are the unique solutions to the congruences

$$c, d \equiv \pm a \pmod{2p}; \quad c, d \equiv \pm b \pmod{2q}.$$

Proof. On $A_{(a,b)}$, the trace of $\rho(x)$ is constant at $2 \cos(a\pi/pq)$, so the same will hold at each endpoint of $A_{(a,b)}$, which corresponds to a reducible representation. But $x \equiv \mu^{\pm q}$ modulo the commutator subgroup, so for any reducible representation ρ we have $\rho(x) = \rho(\mu)^{\pm q}$. If z is a complex q -th root of $\cos(a\pi/pq) \pm i \sin(a\pi/pq)$, then $z = \cos(c\pi/pq) + i \sin(c\pi/pq)$ where $c \equiv \pm a \pmod{2p}$. Hence, for a reducible representation ρ at an endpoint of $A_{(a,b)}$, the trace of $\rho(\mu)$ must be $2 \cos(c\pi/pq)$ with $c \equiv \pm a \pmod{2p}$.

A similar analysis using $\rho(y) = \rho(\mu)^p$ gives the congruence $c \equiv \pm b \pmod{2q}$.

Finally, note that, since $a \equiv b \pmod{2}$ and since p, q are coprime, each of the four pairs of simultaneous congruences

$$c \equiv \pm a \pmod{2p}; \quad c \equiv \pm b \pmod{2q}$$

has a unique solution modulo $2pq$. Moreover, if c is the solution of one of these pairs of congruences, then $2pq - c$ is the solution of another, so precisely two of the four solutions lie in the indicated range $\{1, \dots, pq-1\}$. \square

Proposition 5.2 *Let γ be the closed curve in \mathcal{X} formed by the arc $\overline{A}_{(a,b)}$ together with the subinterval $[2 \cos(c\pi/pq), 2 \cos(d\pi/pq)]$ of $[-2, 2] \cong \mathcal{X}_{red}$. Then $\nu([\gamma], \zeta) \neq 0$.*

Proof. The knot is embedded in an unknotted torus $T \subset S^3$. Each component of $S^3 \setminus T$ is an open solid torus. Moreover, x, y are represented by the cores of these solid tori, and $\zeta = x^p = y^q$ represents a curve on T parallel to the knot. In particular, $\zeta \in A$, ie ζ is a peripheral curve. Now $\rho(\zeta) = \pm I$ for any irreducible representation ρ , and so $\rho(\zeta)$ is constant for $\rho \in A_{(a,b)}$.

Let $z = \exp(i\pi/pq)$, a primitive $(2pq)$ -th root of unity. Then the endpoints of $A_{(a,b)}$ correspond to the reducible representations $\mu \mapsto z^c$ and $\mu \mapsto z^d$, where c, d are given by Lemma 5.1.

Now, as ρ moves continuously through reducible representations from $\mu \mapsto z^c$ to $\mu \mapsto z^d$, the argument of $\rho(\mu)$ changes by $(d - c)\pi/pq$, so the argument of $\rho(\zeta) = \rho(\mu)^{pq}$ changes by $(d - c)\pi$, whence $\nu([\gamma], \zeta) = (d - c)/2 \neq 0$. \square

Corollary 5.3 *Let X be the exterior of a torus knot in S^3 , and $X(\alpha)$ the manifold obtained from X by Dehn filling along a non-meridian slope α . Then $\pi_1(X(\alpha))$ admits a nontrivial representation to $SU(2)$.*

Proof. If γ is the curve in the Theorem, then $\nu([\gamma], \zeta) \neq 0$, and so the kernel of the homomorphism $A \rightarrow \mathbb{Z}$, $\beta \mapsto \nu([\gamma], \beta)$, is precisely $\mu\mathbb{Z}$. But by hypothesis $\alpha \notin \mu\mathbb{Z}$, so $\nu([\gamma], \alpha) \neq 0$. Thus the closed curve $t \mapsto \gamma_t(\alpha)$ has nonzero winding number on S^1 , so is surjective. There is a representation $\rho \in \gamma$ such that $\rho(\alpha) = 1$ in $SU(2)$. This choice of ρ induces a nontrivial representation $\pi_1(X(\alpha)) \rightarrow SU(2)$. \square

Of course, the above corollary is neither new nor surprising. For example, almost all the groups $\pi_1(X(\alpha))$ have nontrivial abelianisation, so admit representations to $SU(2)$ that are reducible but nontrivial. Of more interest is the question of which $\pi_1(X(\alpha))$ admit irreducible representations to $SU(2)$. This question can also be readily answered using the known classification of 3-manifolds obtained by Dehn surgery on torus knots [7]. Here we present an alternative approach using an adaptation of our winding-number technique.

Theorem 5.4 *Let X be the exterior of the (p, q) torus knot, where $1 < p < q$, and $X(\alpha)$ the manifold obtained from X by Dehn filling along a non-meridian slope $\alpha \in \mathbb{Q} \cup \{\infty\}$. Then*

- (1) *if $\alpha = pq$ and $p > 2$, then $\pi_1(X(\alpha))$ admits an irreducible representation to $SU(2)$;*

- (2) if $\alpha = pq$ and $p = 2$, then $\pi_1(X(\alpha))$ admits no irreducible representation to $SU(2)$, but admits a representation to $SO(3)$ with nonabelian image;
- (3) if $\alpha = pq \pm \frac{1}{n}$ for some positive integer n , then every representation from $\pi_1(X(\alpha))$ to $SO(3)$ has abelian image;
- (4) for any other value of α , $\pi_1(X(\alpha))$ admits an irreducible representation to $SU(2)$.

Remark The statement of this theorem fits the classification of [7], where it is proved that $X(\alpha)$ is a lens space in Case (3); a connected sum of two lens spaces in Cases (1) and (2); and a Seifert fibre space in Case (4).

Proof.

(1) Since $2 < p < q$, one of the components of \mathcal{X}_{irr} is the arc $A_{(2,2)}$. But any point on $A_{(2,2)}$ corresponds to a representation ρ with $\rho(x^p) = \rho(y^q) = I$.

(2) In this case $\pi_1(X(\alpha)) \cong \mathbb{Z}_2 * \mathbb{Z}_q$. Since the only element of order 2 in $SU(2)$ is the central element $-I$, the image of any representation $\mathbb{Z}_2 * \mathbb{Z}_q \rightarrow SU(2)$ is abelian. However, corresponding to any point on $A_{(1,1)}$ is a representation ρ with $\rho(x^2) = \rho(y^q) = -I$, so composing this with the quotient map $SU(2) \rightarrow SO(3)$ gives a representation of $\pi_1(X(\alpha))$ to $SO(3)$ with nonabelian image.

(3) Let ζ be the curve $x^p = x^q$ of slope pq . Then $\zeta = \mu^{pq}\lambda$, so $\alpha = \mu^{npq \pm 1}\lambda^n = \mu^{\pm 1}\zeta^n$ in $\pi_1(\partial X)$. Now any representation from $\pi_1(X(\alpha))$ to $SO(3)$ with nonabelian image arises from a representation of $\pi_1(X)$ with nonabelian image, which therefore lifts to an irreducible representation $\rho : \pi_1(X) \rightarrow SU(2)$, such that $\rho(\alpha) = \pm I$. But ρ corresponds to a point on one of the open arcs $A_{(a,b)}$, so $\rho(\zeta) = (-I)^a$ and hence $\rho(\mu) = (\rho(\alpha)\rho(\zeta)^{-n})^{\pm 1} = \pm I$, contradicting the assumption that ρ is irreducible.

(4) As in the previous case, let $\zeta = \mu^{pq}\lambda$ denote the curve with slope pq . Then $\pi_1(\partial X)$ is generated by ζ and μ , so we can write $\alpha = \mu^g\zeta^h$. If $|g| \leq 1$ then we are in one of the previous cases, so we have $|g| \geq 2$.

Suppose first that pq is even. Then the endpoints of $A_{1,1}$ are reducible representations ρ in which the trace of $\rho(\mu)$ is $\pm 2\cos(\pi/pq)$. Choose $\theta \in [\pi/pq, (pq-1)\pi/pq]$ such that θ is an odd multiple of $\pi/|g|$. Then by continuity of trace, we can choose $\rho \in A_{(1,1)}$ such that the trace of $\rho(\mu)$ is $2\cos(\theta)$. Provided h is odd, this gives $\rho(\mu)^g = -I = \rho(\zeta)^{-h}$, so $\rho(\alpha) = I$. If h is even

then $|g|$ is odd, since α is a slope. In particular $|g| > 2$. In this case, we take θ to be an even multiple of $\pi/|g|$, and the argument goes through as before.

Now consider the case where pq is odd. Precisely one of the two positive integers $(q \pm p)/2$ is odd. Call it c , and note that $c \in \{1, \dots, q-1\}$. Let a be the unique odd integer with $1 \leq a \leq p-1$ and $a \equiv \pm c \pmod{p}$. Then the endpoints of $A_{a,c}$ are reducible representations ρ where the trace of $\rho(\mu)$ is $2 \cos(c\pi/pq)$ and $2 \cos((pq - q + c)\pi/pq)$ respectively. Now the interval $[c\pi/pq, (pq - q + c)\pi/pq]$ contains at least one odd multiple of $\pi/|g|$, and (if $|g| > 2$) at least one even multiple of $\pi/|g|$. Arguing as before, we can choose $\rho \in A_{a,c}$ such that $\rho(\mu)^g = \rho(\zeta)^{-h}$, and so $\rho(\alpha) = I$, except possibly if $|g| = 2$ and h is even (which does not arise, since α is a slope). □

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